Problem 3.20

Test the energy-time uncertainty principle for the wave function in Problem 2.5 and the observable x, by calculating σ_H, σ_x, and $d\langle x\rangle/dt$ exactly.

Solution

The energy-time uncertainty principle states that

$$\Delta E\,\Delta t \geq \frac{\hbar}{2}.$$

Write ΔE and Δt in terms of σ_H, σ_x, and $d\langle x\rangle/dt$ with definitions.

$$\sigma_H \frac{\sigma_x}{\left|\frac{d\langle x\rangle}{dt}\right|} \geq \frac{\hbar}{2}$$

Use the definition of σ, the standard deviation.

$$\boxed{\sqrt{\langle H^2\rangle - \langle H\rangle^2}\,\frac{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}{\left|\frac{d\langle x\rangle}{dt}\right|} \geq \frac{\hbar}{2}}$$

The aim now is to calculate all five quantities on the left side for a particle in the infinite square well with position-space wave function,

$$\Psi(x,t) = \frac{1}{\sqrt{a}}e^{-i\omega t}\sin\frac{\pi x}{a} + \frac{1}{\sqrt{a}}e^{-4i\omega t}\sin\frac{2\pi x}{a}, \quad 0 \leq x \leq a,$$

where $\omega = \hbar\pi^2/2ma^2$. Writing it in terms of the eigenstates,

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right)e^{-i\omega t} + \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\right)e^{-4i\omega t} \\ &= \frac{1}{\sqrt{2}}\psi_1(x)e^{-i\omega t} + \frac{1}{\sqrt{2}}\psi_2(x)e^{-4i\omega t},\end{aligned}$$

we see that the probabilities of measuring

$$\begin{aligned}\frac{E_1}{\hbar} &= \omega \quad &\rightarrow \quad E_1 &= \frac{\hbar^2\pi^2}{2ma^2} \\ \frac{E_2}{\hbar} &= 4\omega \quad &\rightarrow \quad E_2 &= \frac{2\hbar^2\pi^2}{ma^2}\end{aligned}$$

are

$$\begin{aligned}P(E_1) &= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \\ P(E_2) &= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},\end{aligned}$$

respectively.

The expectation values involving energy are then

$$\langle H\rangle = P(E_1)E_1 + P(E_2)E_2 = \frac{1}{2}\left(\frac{\hbar^2\pi^2}{2ma^2}\right) + \frac{1}{2}\left(\frac{2\hbar^2\pi^2}{ma^2}\right) = \frac{5\hbar^2\pi^2}{4ma^2}$$

$$\langle H^2\rangle = P(E_1)E_1^2 + P(E_2)E_2^2 = \frac{1}{2}\left(\frac{\hbar^2\pi^2}{2ma^2}\right)^2 + \frac{1}{2}\left(\frac{2\hbar^2\pi^2}{ma^2}\right)^2 = \frac{17\hbar^4\pi^4}{8m^2a^4}.$$

Determine the expectation value of position at time t.

$$\begin{aligned}
\langle x\rangle &= \langle \Psi \,|\, \hat{x} \,|\, \Psi\rangle \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t)\,dx \\
&= \int_0^a \left(\frac{1}{\sqrt{a}}e^{-i\omega t}\sin\frac{\pi x}{a} + \frac{1}{\sqrt{a}}e^{-4i\omega t}\sin\frac{2\pi x}{a}\right)^* (x)\left(\frac{1}{\sqrt{a}}e^{-i\omega t}\sin\frac{\pi x}{a} + \frac{1}{\sqrt{a}}e^{-4i\omega t}\sin\frac{2\pi x}{a}\right)dx \\
&= \frac{1}{a}\int_0^a x\left(\sin^2\frac{\pi x}{a} + 2\cos 3\omega t\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a} + \sin^2\frac{2\pi x}{a}\right)dx \\
&= \frac{1}{a}\left(\int_0^a x\sin^2\frac{\pi x}{a}\,dx + 2\cos 3\omega t\int_0^a x\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\,dx + \int_0^a x\sin^2\frac{2\pi x}{a}\,dx\right) \\
&= \frac{1}{a}\left\{\int_0^a \frac{x}{2}\left(1-\cos\frac{2\pi x}{a}\right)dx + 2\cos 3\omega t\int_0^a \frac{x}{2}\left[\cos\left(\frac{\pi x}{a}-\frac{2\pi x}{a}\right) - \cos\left(\frac{\pi x}{a}+\frac{2\pi x}{a}\right)\right]dx \right.\\
&\qquad\left. + \int_0^a \frac{x}{2}\left(1-\cos\frac{4\pi x}{a}\right)dx\right\} \\
&= \frac{1}{2a}\left[\int_0^a x\,dx - \int_0^a x\cos\frac{2\pi x}{a}\,dx + 2\cos 3\omega t\left(\int_0^a x\cos\frac{\pi x}{a}\,dx - \int_0^a x\cos\frac{3\pi x}{a}\,dx\right)\right.\\
&\qquad\left. + \int_0^a x\,dx - \int_0^a x\cos\frac{4\pi x}{a}\,dx\right] \\
&= \frac{1}{2a}\left[\frac{a^2}{2} - 0 + 2\cos 3\omega t\left(-\frac{2a^2}{\pi^2} + \frac{2a^2}{9\pi^2}\right) + \frac{a^2}{2} - 0\right] \\
&= \frac{1}{2a}\left(a^2 - \frac{32a^2}{9\pi^2}\cos 3\omega t\right) \\
&= \frac{a}{2} - \frac{16a}{9\pi^2}\cos 3\omega t \\
&= \frac{a}{2} - \frac{16a}{9\pi^2}\cos\frac{3\hbar\pi^2}{2ma^2}t
\end{aligned}$$

Differentiate it to get $d\langle x\rangle/dt$.

$$\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\left(\frac{a}{2} - \frac{16a}{9\pi^2}\cos\frac{3\hbar\pi^2}{2ma^2}t\right) = \frac{8\hbar}{3ma}\sin\frac{3\hbar\pi^2}{2ma^2}t$$

Finally, determine the expectation value of x^2 at time t.

$$
\begin{aligned}
\langle x^2\rangle &= \langle \Psi \,|\, \hat{x}^2 \,|\, \Psi\rangle \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t)(x^2)\Psi(x,t)\,dx \\
&= \int_0^a \left(\frac{1}{\sqrt{a}}e^{-i\omega t}\sin\frac{\pi x}{a} + \frac{1}{\sqrt{a}}e^{-4i\omega t}\sin\frac{2\pi x}{a}\right)^* (x^2)\left(\frac{1}{\sqrt{a}}e^{-i\omega t}\sin\frac{\pi x}{a} + \frac{1}{\sqrt{a}}e^{-4i\omega t}\sin\frac{2\pi x}{a}\right) dx \\
&= \frac{1}{a}\int_0^a x^2\left(\sin^2\frac{\pi x}{a} + 2\cos 3\omega t\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a} + \sin^2\frac{2\pi x}{a}\right) dx \\
&= \frac{1}{a}\left(\int_0^a x^2\sin^2\frac{\pi x}{a}\,dx + 2\cos 3\omega t\int_0^a x^2\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\,dx + \int_0^a x^2\sin^2\frac{2\pi x}{a}\,dx\right) \\
&= \frac{1}{a}\left\{\int_0^a \frac{x^2}{2}\left(1-\cos\frac{2\pi x}{a}\right)dx + 2\cos 3\omega t\int_0^a \frac{x^2}{2}\left[\cos\left(\frac{\pi x}{a}-\frac{2\pi x}{a}\right) - \cos\left(\frac{\pi x}{a}+\frac{2\pi x}{a}\right)\right]dx \right.\\
&\qquad\left. + \int_0^a \frac{x^2}{2}\left(1-\cos\frac{4\pi x}{a}\right)dx\right\} \\
&= \frac{1}{2a}\left[\int_0^a x^2\,dx - \int_0^a x^2\cos\frac{2\pi x}{a}\,dx + 2\cos 3\omega t\left(\int_0^a x^2\cos\frac{\pi x}{a}\,dx - \int_0^a x^2\cos\frac{3\pi x}{a}\,dx\right)\right.\\
&\qquad\left. + \int_0^a x^2\,dx - \int_0^a x^2\cos\frac{4\pi x}{a}\,dx\right] \\
&= \frac{1}{2a}\left[\frac{a^3}{3} - \frac{a^3}{2\pi^2} + 2\cos 3\omega t\left(-\frac{2a^3}{\pi^2} + \frac{2a^3}{9\pi^2}\right) + \frac{a^3}{3} - \frac{a^3}{8\pi^2}\right] \\
&= \frac{1}{2a}\left(\frac{2a^3}{3} - \frac{5a^3}{8\pi^2} - \frac{32a^3}{9\pi^2}\cos 3\omega t\right) \\
&= \frac{a^2}{3} - \frac{5a^2}{16\pi^2} - \frac{16a^2}{9\pi^2}\cos 3\omega t \\
&= \frac{a^2}{3} - \frac{5a^2}{16\pi^2} - \frac{16a^2}{9\pi^2}\cos\frac{3\hbar\pi^2}{2ma^2}t
\end{aligned}
$$

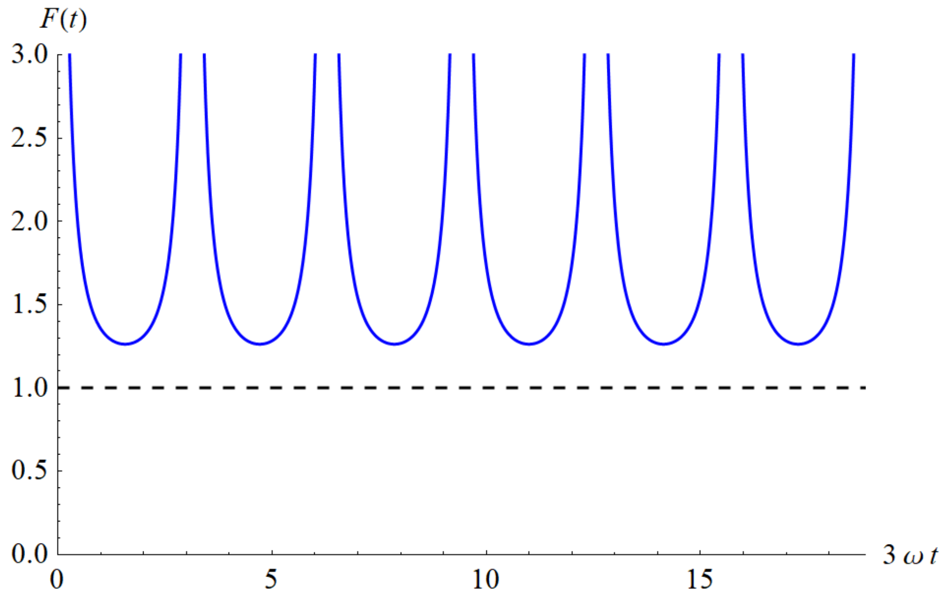
The left side of the boxed formula evaluates to

$$
\begin{aligned}
\sqrt{\langle H^2\rangle - \langle H\rangle^2}\,\frac{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}{\left|\frac{d\langle x\rangle}{dt}\right|} &= \sqrt{\left(\frac{17\hbar^4\pi^4}{8m^2a^4}\right) - \left(\frac{5\hbar^2\pi^2}{4ma^2}\right)^2}\,\frac{\sqrt{\left(\frac{a^2}{3} - \frac{5a^2}{16\pi^2} - \frac{16a^2}{9\pi^2}\cos\frac{3\hbar\pi^2}{2ma^2}t\right) - \left(\frac{a}{2} - \frac{16a}{9\pi^2}\cos\frac{3\hbar\pi^2}{2ma^2}t\right)^2}}{\left|\frac{8\hbar}{3ma}\sin\frac{3\hbar\pi^2}{2ma^2}t\right|} \\
&= \frac{3\hbar^2\pi^2}{4ma^2}\,\frac{\sqrt{\frac{a^2}{12} - \frac{5a^2}{16\pi^2} - \frac{256a^2}{81\pi^4}\cos^2\frac{3\hbar\pi^2}{2ma^2}t}}{\frac{8\hbar}{3ma}\left|\sin\frac{3\hbar\pi^2}{2ma^2}t\right|} \\
&= \frac{\hbar}{2}\left(\frac{9\pi^2}{16}\sqrt{\frac{\frac{1}{12} - \frac{5}{16\pi^2} - \frac{256}{81\pi^4}\cos^2\frac{3\hbar\pi^2}{2ma^2}t}{\sin^2\frac{3\hbar\pi^2}{2ma^2}t}}\right).
\end{aligned}
$$

Let

$$F(t) = \frac{9\pi^2}{16}\sqrt{\frac{\frac{1}{12} - \frac{5}{16\pi^2} - \frac{256}{81\pi^4}\cos^2 3\omega t}{\sin^2 3\omega t}},$$

and plot $F(t)$ versus $3\omega t$. For the energy-time uncertainty principle to be satisfied, the graph must be greater than or equal to 1 for all time.



And it is indeed.